

Dynamical Symmetry Breaking in QED_3 from the Wilson RG Point of View

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Abstract

Dynamical symmetry breaking in three dimensional QED with N flavors, which has been mostly analyzed by solving the Schwinger-Dyson equations, is investigated by means of the approximated Wilson, or non-perturbative, renormalization group (RG). We study the RG flows of the gauge coupling and the general four-fermi couplings allowed by the symmetry with concentrating our interest on study of the phase structure. The RG equations have no gauge parameter dependence in our approximation scheme. It is found that there exist chirally broken and unbroken phases for $N > N_{\text{cr}}$ ($3 < N_{\text{cr}} < 4$) and that the unbroken phase disappears for $N < N_{\text{cr}}$. We also discuss the spontaneous parity breaking in QED_3 with the four-fermi interactions.

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§1. Introduction

Dynamical symmetry breaking in (2+1)-dimensional massless Quantum Electrodynamics (QED₃) has been attracting much attention since Pisarski and Appelquist *et al.*^{1), 2)} have found the novel chiral symmetry breaking depending on the number of flavors by solving the approximated Schwinger-Dyson (SD) equations. In the case that QED₃ contains N flavors of four-component spinors, or $2N$ flavors of two-component spinors, the global symmetry is enhanced to $U(2N)$ rather than $U(N)$. This enhanced symmetry may be regarded as a sort of chiral symmetry even though the theory is defined in three-dimensions. Indeed this symmetry is spontaneously broken by dynamical mass generation.

In these analyses the photon self-energy evaluated in the large N limit is inserted into the gap equation obtained in the so-called ladder approximation. It has been claimed in this way²⁾ that the chiral symmetry is spontaneously broken for $N < N_{\text{cr}} = 32/\pi^2 \sim 3.24$, while unbroken for $N > N_{\text{cr}}$. In practice the SD equations derived in the ladder approximation suffer from large gauge parameter dependence. The above result is brought by using the Landau gauge.

Subsequently a lot of works have been devoted for improvement of the approximation schemes for the SD equations and for studies of various aspects of the phase transition^{3) - 5)}. In ref. 3), the approximation is refined so that the gap equations is made gauge parameter independent by proceeding to the next-to-leading order of $1/N$ expansion. Later the effects of the wave function renormalization ignored in the original approximation have been also incorporated. Then the coupled SD equations for the photon self-energy and the mass function were examined. As results, such elaborate treatment of the SD equations also supports the qualitative picture of the chiral symmetry breaking mentioned above. Even the critical flavor number has not been altered significantly.

On the other hand numerical simulations have been also performed for non-compact QED₃ defined on the lattice⁶⁾. It seems remarkable that the results from these simulations are consistent with those obtained by solving the approximated SD equations, namely $3 < N_{\text{cr}} < 4$.

QED₃ is invariant also under parity transformation as long as the Chern-Simons term is absent. It has been shown by Redlich⁷⁾ that the Chern-Simons term should be generated through gauge invariant regularization with odd number of two-component massless spinors. However it is expected according to Vafa-Witten's argument⁸⁾ that the parity symmetry is not spontaneously broken for QED₃ with four-component spinors. Actually it was seen⁹⁾ that the analyses using the SD equations are consistent with this expectation by considering the gap equations for the chirally invariant but parity odd dynamical mass.

After that these studies were extended to QED_3 with fermion self-interactions¹⁰⁾. This was motivated by the expectation that the parity symmetry can be spontaneously broken in the presence of general four-fermi interactions. The SD analyses show the presence of parity broken phase and also suggest that this phase is separated from the chiral symmetry broken phase. The Chern-Simons term is generated through radiative correction in the parity broken phase even in the case of even number of flavors.

Apart from this, the dynamics of QED_3 with the Chern-Simons term has been also intensively discussed^{11), 12)}, specially in view of its applications in modelling (high-temperature) superconductors^{13), 14)}. In this paper, however, we consider only QED_3 with even number of flavors and do not include the Chern-Simons term. Furthermore the three dimensional Thirring model were also examined by numerical simulations as well as by solving the SD equations¹⁵⁾.

The Exact Renormalization Group (ERG)^{16) - 20)}, which represents continuous version of the Wilson RG transformation, has been known as an analytical method applicable to non-perturbative dynamics of field theories. Both of the ERG and the SD equations are given in functional forms and lead to the correlation functions as their solutions. It is also common that regularization is necessary and that some approximations are inevitable in practical calculations, but it should be noted that the ERG equations give the RG flows for the effective couplings, while the SD gives order parameters in terms of bare parameters.^{*)} In analysis shown in this paper, we will perform rather brute approximations. Therefore let us call the approximated ERG as Non-perturbative (NP) RG hereafter.

NPRG has not been applied to clarify the phase structures of dynamical symmetry breaking in QED_3 (with the four-fermi interactions) in spite of much interest mentioned above.

^{**) The purpose of this paper is to show that we may clarify the phase structures of these theories much easier than the SD approaches. It will be clearly seen by NPRG that presence of the IR attractive fixed point is essential for the novel phase transition. Moreover we can directly find out the boundaries between the chiral symmetry broken phase, the parity broken phase and the unbroken phase just by following the RG flows. This point is a great contrast to the SD approach. Note that in the SD approach we need to assume the order parameters for the symmetries to be broken apriori to derive the gap equations and to solve them for every theory. In the RG approach, however, we may treat any theories invariant under the symmetries on the equal footing without concerning the order parameters. We aim to demonstrate such advantageous points of NPRG in comparison with the}

^{*)} The relations between ERG and SD equations are discussed in ref. 21).

^{**) In ref. 22) dynamical symmetry breaking of four-fermi theories are examined by applying the RG method to their effective (composite) theories.}

SD analyses through explicit calculations for QED_3 , not to pursue further improvement of the approximations.

Dynamical chiral symmetry breaking in $(3+1)$ -dimensional gauge theories has been analyzed by NPRG in the case of a single massless flavor²³⁾⁻²⁵⁾. Four-fermi interactions are induced in the effective theories by the gauge interactions. It was found that the RG flows of the four-fermi couplings reveal the phase structure. Also if we truncate the corrections for the four-fermi interactions up to a set called the ladder-type in the later discussions, then the phase boundary obtained by solving the SD equation in the ladder approximation is found to be precisely reproduced. The anomalous dimensions of fermion composite operators are immediately calculated from the RG equations.

Moreover the gauge parameter dependence is remarkably improved by using the NPRG equations obtained by the derivative expansion²³⁾. The momentum cutoff introduced to define the Wilson RG breaks the gauge invariance. The remained gauge dependence comes purely from this cutoff procedure, while the ladder approximation badly destroys the gauge invariance. We may remove the gauge dependence by substituting the anomalous dimension of the fermion with one evaluated in a gauge invariant way. We will adopt such an approximation scheme in the RG analyses of QED_3 as well.

The rest of our paper is organized as follows. In section 2, we briefly review the NPRG for a simple four-fermi theory and how dynamical symmetry breaking is described in the RG framework. Section 3 is our main part. First we consider the general four-fermi operators allowed by the symmetries of QED_3 . After explaining our approximation scheme to the NPRG, we explicitly examine the phase structure by looking at the RG flows with varying the number of flavors. In the Wilson RG framework we treat all the effective theories at the same time. In our scheme the phases of QED_3 with general four-fermi interactions are explored automatically. Thus we can discuss not only the chiral symmetry breaking but also the parity breaking. Lastly the broken symmetries are discussed in section 4. Actually the RG flows themselves tell about the phases but not the symmetries. We discuss a simple way to know the broken symmetry by utilizing the RG flows. Section 5 is devoted to conclusions.

§2. RG equation for the four-fermi coupling and chiral symmetry breaking

Here we consider the large N Gross-Neveu model as the simplest example for the dynamical chiral symmetry breaking^{23), 24)}. Of course this model can be analyzed easily by considering the gap equation. Let us see how we can obtain the same results by solving the NPRG equations.

Wilson RG observes variation of the effective actions in lowering the cutoff scale. Since

any operators allowed by the original symmetries are generated in the effective action through the radiative corrections, we need to consider infinitely many effective couplings. In the large N limit, however, the ERG equation for the cutoff effective potential $V(\psi, \bar{\psi}; \Lambda)$ may be exactly derived. The approximation truncating any momentum dependent corrections is called the Local Potential Approximation (LPA)²⁶⁾. Namely the LPA becomes exact in the large N limit.

In the later discussion we want to compare the results with those obtained by the SD equations defined with sharp cutoff. Therefore let us adopt the sharp cutoff also for the RG equations. It has been also known²⁷⁾ that the sharp cutoff limit of ERG equations reduce to the Wegner-Houghton RG equation¹⁷⁾. In this formulation cutoff is performed to the path integration measure:

$$Z = \int \prod_{|p| < \Lambda} \mathcal{D}\psi(p) \mathcal{D}\bar{\psi}(p) e^{-S_{\text{eff}}[\psi, \bar{\psi}; \Lambda]}. \quad (2.1)$$

Here we may set the euclidian effective action for the large N Gross-Neveu model in D dimensions as

$$S_{\text{eff}} = \int d^D x \bar{\psi}_i \gamma_\mu \partial_\mu \psi_i + V(\sigma), \quad (2.2)$$

where σ denotes a product of the classical fields, $\bar{\psi}\psi$. It should not be confused with the expectation value of the fermion composite, $\langle \bar{\psi}\psi \rangle$.

The ERG for the effective potential is given with the scale parameter $t = \ln(\Lambda_0/\Lambda)$ by

$$\frac{dV(\sigma)}{dt} = DV(\sigma) - (D-1)V'(\sigma) - A_D \ln(1 + V'(\sigma)^2), \quad (2.3)$$

where the prime stands for the derivative with respect to σ and A_D is a constant depending on the space-time dimensions. It is found²⁴⁾ that the dynamical mass treated by the SD equations is obtained from this effective potential as

$$m_{\text{eff}} = \lim_{\Lambda \rightarrow 0} V'(\sigma, \Lambda)|_{\sigma=0}. \quad (2.4)$$

The ERG has a great advantage to find the phase structures and also the critical exponents compared with the SD approaches. If we perform the operator expansion of the effective potential into

$$V(\sigma; \Lambda) = -\frac{1}{2\Lambda^{D-2}}G(\Lambda)\sigma^2 + \frac{1}{8\Lambda^{3D-4}}G_8(\Lambda)\sigma^4 + \dots, \quad (2.5)$$

we may derive beta functions for each coupling. Then the effective four-fermi coupling G is found to be subject to the ERG equation isolated from others:

$$\beta_G = \Lambda \frac{dG}{d\Lambda} = (D-2)G - A_D G^2. \quad (2.6)$$

This beta function has two fixed points: $G^* = 0$ (IR attractive) and $G^* = (d-2)/A_D$ (IR repulsive). The IR repulsive fixed point gives the critical coupling of the chiral symmetry breaking. Thus we can immediately find from the RG flows that there exist two phases; broken and unbroken ones. It is also quite easy to calculate the anomalous dimensions of the operators $\bar{\psi}\psi$, $(\bar{\psi}\psi)^2$ and so on by this method.

It would be important to note that the mass term or any symmetry breaking operator does not appear in the effective action even in the chirally broken phase since the RG evolution respects the original symmetries. However if we solve the ERG equation for the effective potential $V(\sigma; \Lambda)$, then it is found that the potential is evolved to be non-analytic at the origin due to IR singularity of massless fermion loops²⁴⁾. Thus the dynamical mass generation is observed in a rather non-trivial way. As is shown in ref. 24), it is practically useful to introduce collective coordinates corresponding to the fermion composites into the effective action in order to evaluate the order parameters. However it should be noted that the RG flows of the four-fermi couplings are unable to conclude the broken symmetries even though the phase boundaries are exposed by them. In section 4 we are coming back to this problem in order to see the symmetry spontaneously broken in QED₃.

§3. NPRG for QED₃ and dynamical symmetry breaking

3.1. Four-fermi interactions

Let us consider QED₃ with N flavors of four-component spinors, ψ^i ($i = 1, \dots, N$). The bare lagrangian is given in euclidian space by

$$\mathcal{L}_b = \frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}_i \gamma_\mu (\partial_\mu - ieA_\mu) \psi^i - \frac{1}{2\xi}(\partial_\mu A_\mu)^2, \quad (3.1)$$

where we suppose that the Chern-Simons term is absent. We use the 4 by 4 γ matrices given by

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (3.2)$$

Also we introduce $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ and $\tau = -i\gamma_5\gamma_3$.

This lagrangian is invariant under the global $U(2N)$ and also the parity symmetry. The parity transformation is defined by $\psi^i(x) \mapsto \psi'^i(x') = i\gamma_3\gamma_1\psi^i(x)$ for $x' = (t, -x, y)$. $U(2N)$ symmetry is made more transparent by reformulating in terms of $2N$ two-component spinors, χ^I ($I = 1, \dots, 2N$):

$$\psi^i = \begin{pmatrix} \chi^i \\ \chi^{i+N} \end{pmatrix}, \quad \bar{\psi}_i = (\chi_i^\dagger \sigma_3, -\chi_{i+N}^\dagger \sigma_3) = (\bar{\chi}_i, \bar{\chi}_{i+N})\tau. \quad (3.3)$$

The two-component fields are transformed by the $U(2N)$ matrix U as $\chi^I \mapsto \chi'^I = U_J^I \chi^J$. Therefore $\bar{\psi}_i \gamma_\mu \psi^i = \bar{\chi}_I \sigma_\mu \chi^I$ is invariant under the both symmetries. The ordinary mass operator, $\bar{\psi}_i \psi^i = \bar{\chi}_i \chi^i - \bar{\chi}_{i+N} \chi^{i+N}$, is parity even but not invariant under the $U(2N)$ transformation. If this operator acquires a non-vanishing vacuum expectation value, then $U(2N)$ is spontaneously broken to $U(N) \times U(N)$. Thus we may regard this $U(2N)$ symmetry as a sort of chiral transformation. While we find a $U(2N)$ invariant operator, $\bar{\psi}_i \tau \psi^i = \bar{\chi}_I \chi^I$, it is parity odd in turn. Therefore non-vanishing expectation value of this operator leads to spontaneous breakdown of the parity symmetry. However it is expected from the Vafa-Witten theorem that the parity symmetry is never broken in QED₃.

In section 2 we saw that the RG flows of the effective four-fermi interactions are important to distinguish the phases. We can list up all the local four-fermi operators invariant under $U(2N)$ and parity transformations as follows:

$$\mathcal{O}_P = (\bar{\psi}_i \tau \psi^i)^2 = (\bar{\chi}_I \chi^I)^2 \quad (3.4)$$

$$\mathcal{O}_V = (\bar{\psi}_i \gamma_\mu \psi^i)^2 = (\bar{\chi}_I \sigma_\mu \chi^I)^2 \quad (3.5)$$

$$\begin{aligned} \mathcal{O}_S &= \frac{1}{2} [\bar{\psi}_i \psi^j \bar{\psi}_j \psi^i - \bar{\psi}_i \gamma_3 \psi^j \bar{\psi}_j \gamma_3 \psi^i - \bar{\psi}_i \gamma_5 \psi^j \bar{\psi}_j \gamma_5 \psi^i + \bar{\psi}_i \tau \psi^j \bar{\psi}_j \tau \psi^i] \\ &= \bar{\chi}_I \chi^J \bar{\chi}_J \chi^I \end{aligned} \quad (3.6)$$

$$\begin{aligned} \mathcal{O}_{V'} &= \bar{\psi}_i \gamma_\mu \psi^j \bar{\psi}_j \gamma_\mu \psi^i - \bar{\psi}_i \gamma_3 \gamma_\mu \psi^j \bar{\psi}_j \gamma_3 \gamma_\mu \psi^i \\ &\quad - \bar{\psi}_i \gamma_5 \gamma_\mu \psi^j \bar{\psi}_j \gamma_5 \gamma_\mu \psi^i + \bar{\psi}_i \tau \gamma_\mu \psi^j \bar{\psi}_j \tau \gamma_\mu \psi^i \\ &= 2 \bar{\chi}_I \sigma_\mu \chi^J \bar{\chi}_J \sigma_\mu \chi^I \end{aligned} \quad (3.7)$$

These operators are induced by radiative corrections. However it is found by the Fierz transformation that only two of them are independent. We choose \mathcal{O}_S and \mathcal{O}_P as the independent ones and always rewrite others by using the Fierz transformation whenever they are induced.

3.2. NPRG and the approximation scheme

In this subsection we explain the outline of our approximation scheme. First we truncate the set of induced operators in the effective lagrangian to

$$\mathcal{L}_{\text{eff}} = \frac{1}{4} F_{\mu\nu}^2 + \bar{\chi}_I (\not{\partial} - ie\not{A}) \chi^I - \frac{1}{2\xi} (\partial_\mu A_\mu)^2 - \frac{G_S}{2} \mathcal{O}_S - \frac{G_P}{2} \mathcal{O}_P. \quad (3.8)$$

Therefore the RG flows are given in the three dimensional coupling space spanned by (e^2, G_S, G_P) . As is seen in the previous section the RG equations of the four-fermi couplings are separated from other multi-fermi couplings in the LPA. Namely we may obtain the same RG equations for the four-fermi couplings if we perform the operator expansion to the effective potential evaluated in the LPA. The reason to truncate other operators is that

the four-fermi couplings are enough to explore the phase structures. In the RG approach we may naturally incorporate all the theories with the identical symmetries. For example, we can consider the model with a bare \mathcal{O}_V operator, namely the massless Thirring model, at the same time.

Since our purpose is to see the chiral phase structure of QED₃, we adopt the chiral $U(2N)$ symmetry and parity preserving regularization, *i.e.* naive momentum cutoff, at the cost of the gauge invariance. Here we discard the gauge non-invariant corrections induced in such a regularization scheme, *e.g.* photon mass, and substitute the beta function for the gauge coupling by the one-loop perturbative one as the first step of approximation. Of course the gauge invariant scheme²⁸⁾ is preferable to see non-perturbative dynamics by gauge interactions in general. However, the manifest chiral symmetry would be necessarily lost and we would face up to the problem to extract chirally invariant theories. This is similar to the problem appearing in the lattice gauge theories. In this paper we do not pursue for this direction.

The Chern-Simons term

$$\mathcal{L}_{\text{CS}} = \frac{i}{2} \theta \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho \quad (3.9)$$

cannot be generated in the Wilsonian effective action since such a correction is forbidden by the parity symmetry. On the other hand, however, in the parity broken phase (which exists for the models with the bare four-fermi interactions), the fermions are supposed to acquire the parity breaking effective mass. Therefore the CS term is expected to be dynamically generated through radiative corrections. This contradicting situation is related to the appearance of the dynamical mass in the RG framework. It is expected that the Chern-Simons term will be generated once we reevaluate the corrections at the parity broken vacua. Here we leave this problem to the future investigations.

Now we consider the RG equations for the three couplings, (e^2, G_s, G_p) . It is not necessary to derive the NPRG for the effective action to find them. Indeed the corrections appearing in the ERG formulation are given also by evaluating one-loop diagrams with internal momentum scale fixed to Λ .

First the beta function for the gauge coupling may be obtained in the above approximation scheme as

$$\frac{de^2}{dt} = e^2 - \frac{N}{8} e^4, \quad (3.10)$$

where $t = \ln(\Lambda_0/\Lambda)$. The first term represents the canonical scaling of the gauge coupling with dimension one half.

In the original SD analyses^{1), 2)} the photon self-energy was evaluated in the large N leading order as $\Pi(p^2) = Ne_0^2/8\sqrt{p^2}$ with the bare gauge coupling e_0 . Note that the self-

energy is finite in three dimensions. The photon propagator in the gap equation is defined with this self-energy.

We may define also the renormalized gauge coupling as

$$e^2(p) = \frac{e_0^2}{1 + \Pi(p^2)}. \quad (3.11)$$

If we identify the renormalization scale μ with the momentum p^2 , then this renormalized coupling satisfies the above RG equation. It should be noted that there appears an IR stable fixed point of the gauge coupling in three dimensions. As is seen later this fixed point plays an essential role for the novel phase transition in the RG point of view. The fixed point coupling is given by $e^{2*} = 8/N$. When N is not large, this appears at the strong coupling region. Therefore the present perturbative treatment is not justified actually. However, as long as the fixed point structure of the gauge beta function is not altered significantly, the essential mechanism of the dynamical symmetry breaking is thought to be captured in this simple RG treatment. The evaluation of the gauge beta function reliable in the strong coupling region remains as an open problem.

The beta functions for the four-fermi couplings G_s and G_P are evaluated by summing up the corrections described in Fig. 1. At the vertices the operators corresponding to the couplings are inserted. We call the diagrams in the first two lines of Fig. 1 “ladder type” and the others “non-ladder type”. When we restrict ourselves “ladder type diagram”, the results obtained in the ladder SD equation are precisely reproduced as we will see below. We examine the RG flows explicitly and discuss the phase structures in the case including only the ladder type corrections and the full corrections separately in the next two subsections.

3.3. The ladder approximation

We also adopt the Landau gauge propagator to evaluate the ladder type diagrams. Then the beta functions for the four-fermi couplings G_s and G_P are easily found out to be

$$\dot{G}_s = -G_s + \frac{1}{\pi^2} \left[G_s^2 - G_s G_P + \frac{1}{3} G_P^2 + 2e^2 G_s - \frac{4}{3} e^2 G_P + \frac{2}{3} e^4 \right], \quad (3.12)$$

$$\dot{G}_P = -G_P + \frac{1}{\pi^2} \left[\frac{1}{6} G_P^2 - \frac{2}{3} e^2 G_P - \frac{2}{3} e^4 \right], \quad (3.13)$$

where dot in the left hand side stands for the derivative with respect to $t = \ln(\Lambda_0/\Lambda)$. By defining a new variable, $G'_s = G_s - G_P/2$, we can separate the beta function for G'_s from that for G_P as

$$\dot{G}'_s = -G'_s + \frac{1}{\pi^2} \left[G'^2_s + 2e^2 G'_s + e^4 \right]. \quad (3.14)$$

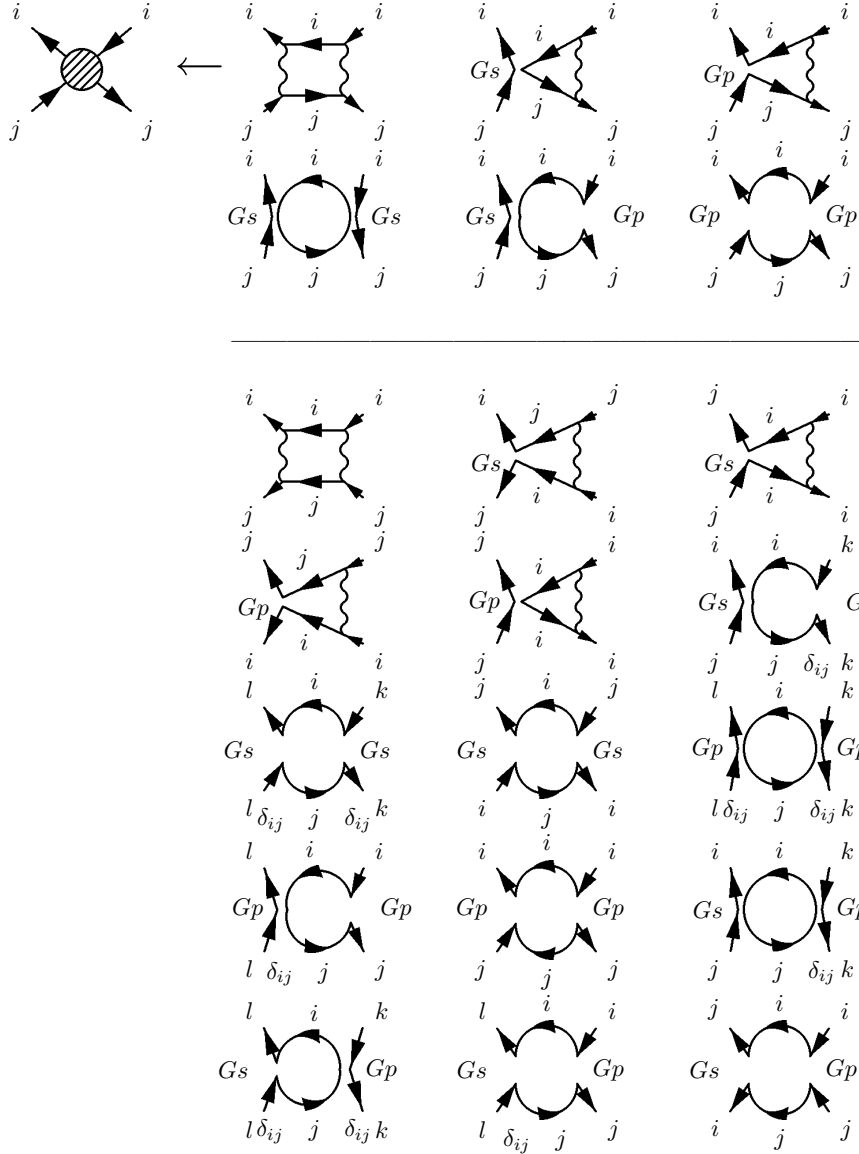


Fig. 1. The Feynman diagrams for the corrections to the four-fermi operators are shown. The arrows represent contraction of the spinor indices.

Now we may solve the coupled equations Eq. (3.10) and Eq. (3.14). It is easily found that there exist two fixed points, $G_s'^{(\pm)}$,

$$G_s'^{(\pm)} = \frac{1}{2} \left(1 - \frac{2e^{2*}}{\pi^2} \right) \pm \sqrt{\frac{1}{4} - \frac{e^{2*}}{\pi^2}}, \quad (3.15)$$

only if $e^{2*} < \pi^2/4$, *i.e.* $N > N_{\text{cr}} = 32/\pi^2$. At the critical number, these two points merge each other and there is no fixed point solution for $N < N_{\text{cr}}$.

$G_s'^{(-)}$ ($G_s'^{+}$) is an IR (UV) fixed point respectively. Existence of the IR fixed point, $G_s'^{(-)}$, is important as far as the dynamical symmetry breaking in QED₃ is concerned. The

RG flow diagrams in (e^2, G'_s) -space are shown in Fig. 2 and Fig. 3 in the case of $N = 4$ and 2 respectively. Note that QED₃ corresponds to flows starting from the $G'_s = 0$ line. In Fig. 2 we see that the IR fixed point exists indeed. In the asymptotically free region, all the flows are absorbed into the IR fixed point, which means the theory is in the symmetric phase. On the other hand, in the asymptotically non-free region, there appears a phase boundary. This indicates that dynamical chiral symmetry breaking occurs even if $N > N_{\text{cr}}$, provided the bare gauge coupling is strong enough. As is seen in Fig. 3, all the flows in both the regions blow up for $N < N_{\text{cr}}$. From this behavior we may suppose that the chiral symmetry is spontaneously broken irrespective of the bare gauge coupling.

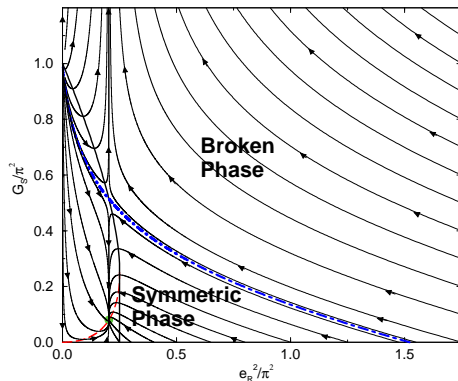


Fig. 2. RG flows for QED₃ with $N = 4$ flavors in the ladder approximation are described on the (e^2, G_s) -plane. The dashed-dot line is the chiral phase boundary.

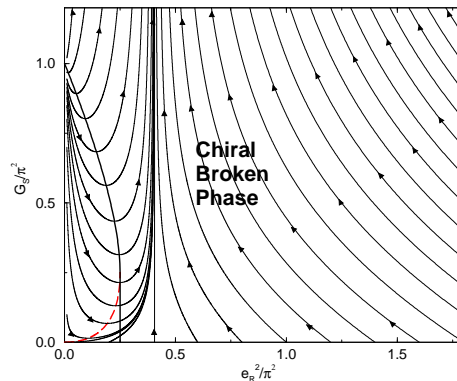


Fig. 3. RG flows for QED₃ with $N = 2$ flavors in the ladder approximation are described on the (e^2, G_s) -plane.

Our conclusion may sound slightly different from the results advocated in ref. 1), though the critical number of flavor just coincides. It was claimed that there is no non-trivial solution of the gap equation as far as $N > N_{\text{cr}}$ no matter how strong the gauge coupling is. The difference is thought to lie in the way of renormalization of the gauge coupling. If we make the renormalized coupling defined by Eq. (3.11) dimensionless by scaling $e^2 \rightarrow e^2 \mu$, then e^2 never exceeds the fixed point value. Namely the SD analysis examined only the asymptotically free region of Fig. 2 and Fig. 3 in the RG point of view. Therefore the SD result is not contradicting with our observation.

In the RG approach we put cutoff for any radiative corrections. The effective gauge coupling obtained by solving Eq. (3.10) is given by

$$e^2(\Lambda) = \frac{e_0^2}{1 + \frac{Ne_0^2}{8\Lambda} - \frac{Ne_0^2}{8\Lambda_0}}. \quad (3.16)$$

This effective coupling can be made bigger than the IR fixed point, e^{2*} , owing to the bare cutoff scale. On the other hand the renormalized coupling defined with the finite self energy corresponds to the infinite limit of the bare cutoff. Namely it may be said that the SD analysis restricts QED₃ which possesses continuum limit. In the lattice simulations as well the continuum limit is observed⁶⁾. Reversely we cannot take the continuum limit of the model with the gauge coupling larger than e^{2*} . However we should be allowed to treat QED₃ as an effective theory with some UV cutoff. Then there are found two phases.

3.4. Results by the gauge independent RG equations

Now let us investigate the RG flows taking account of the full corrections to the four-fermi couplings given by Fig. 1. Both of the contributions from the ladder diagrams and from the non-ladder diagrams amount to the same order. Therefore we should include the non-ladder ones as well. However the most benefit to include them is recovering the gauge independence lost in the ladder approximation. The gauge parameter in the e^4 order contributions are found to cancel each other. The gauge parameter dependence in the $e^2 G_s(G_P)$ order should be eliminated by considering the fermion anomalous dimensions η ²³⁾. Since the anomalous dimension cannot be evaluated in the LPA, we must proceed to the derivative expansion. Here, however, we simply substitute the perturbative result for η :

$$\eta = - \left(\frac{16 - 3N\xi}{6N\pi^2} \right) e^2. \quad (3.17)$$

Then the beta functions for G_s and G_P are found to be

$$\dot{G}_s = -G_s + \frac{1}{\pi^2} \left[\frac{N+2}{3} G_s^2 - G_s G_P + \frac{4}{3} e^2 G_s - \frac{8}{3} e^2 G_P \right], \quad (3.18)$$

$$\begin{aligned} \dot{G}_P = -G_P + \frac{1}{\pi^2} \left[(2N-1) G_P^2 - 2(N-1) G_s G_P + \frac{4N-7}{6} G_s^2 \right. \\ \left. - \frac{8}{3} e^2 G_s + \frac{4}{3} e^2 G_P - 2e^4 \right], \end{aligned} \quad (3.19)$$

which are free from the gauge parameter.

In this case we cannot separate G_s part by the redefinition. Therefore we examine these coupled differential equations including Eq. (3.10) numerically. The RG flows run in the three dimensional theory space. In Fig. 3 and Fig. 4 the RG flows for QED₃, or starting from $G_s = G_P = 0$, are projected on the (e^2, G_s) -plane. Fig. 4 is for $N = 4$ and Fig. 5 is for $N = 2$. Comparing Fig. 2 and Fig. 3 with Fig. 4 and Fig. 5, we notice that there is no qualitative difference between them. This critical number of flavor is also evaluated by the numerical calculation and found to be $N_{cr} \simeq 3.1$. This RG analysis naively indicates that QED₃ becomes a scale invariant theory, if the flows are absorbed into the IR fixed point. This is possible only for $N > N_{cr}$.

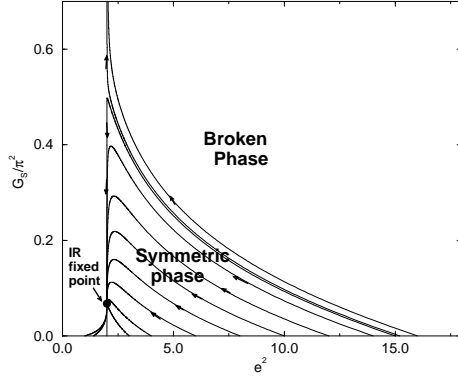


Fig. 4. RG flows for QED₃ with $N = 4$ flavors in the LPA are projected on the (e^2, G_S) -plane.

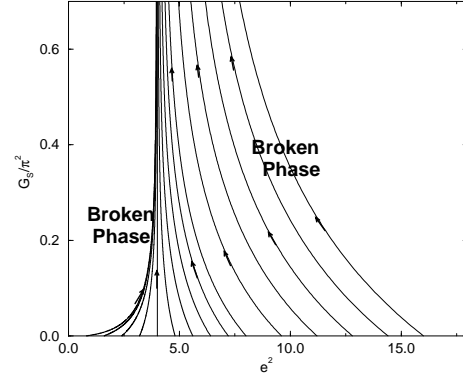


Fig. 5. RG flows for QED₃ with $N = 2$ flavors in the LPA are projected on the (e^2, G_S) -plane.

3.5. Parity broken phase

In Fig. 6 and Fig. 7, the RG flows on the IR fixed point for the gauge coupling are shown in (G_P, G_S) -plane in the case of $N = 4$ and 2 respectively. It is seen that there are three distinct phases for $N = 4$, while the symmetric phase, where the flows are absorbed into the IR fixed point, vanishes for $N = 2$. The transition from three phases to two phases also occurs at $N = N_{\text{cr}}$. The chiral symmetry is supposed to be broken in the upper phase and the parity symmetry is broken in the right phase. We cannot conclude the broken symmetries from these RG flows. However we may find the good evidence by the argument discussed in the next section.

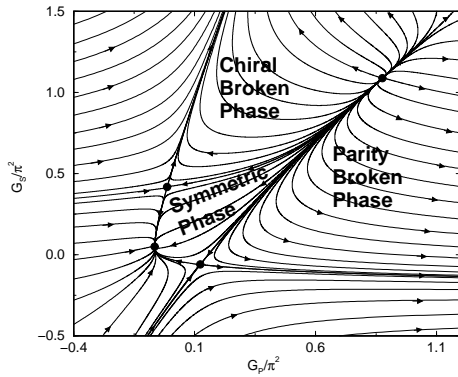


Fig. 6. RG flows for QED₃ with $N = 4$ flavors in the LPA are projected on the (G_P, G_S) -plane.

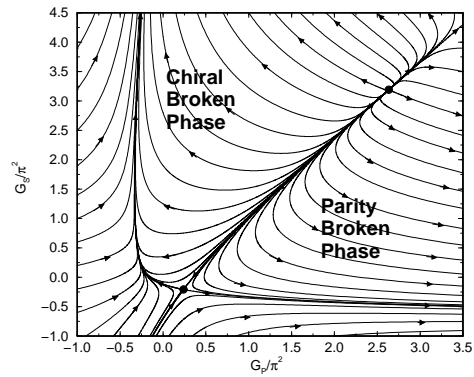


Fig. 7. RG flows for QED₃ with $N = 2$ flavors in the LPA are projected on the (G_P, G_S) -plane.

Indeed the RG flows for QED₃ do not enter the parity broken phase. This corresponds

to the Vafa-Witten theorem. However the phase structure found here shows that the parity symmetry can be spontaneously broken for the models with the bare four-fermi interactions. This also coincides with the results obtained by the SD methods¹⁰⁾. *) Note that we find a tri-critical fixed point at the edge of the phase boundary between the chiral broken phase and the parity broken phase. The tri-criticality indicates that the transition between these two phases becomes first order beyond this edge. This point seems to deserve for further study in the NPRG point of view. Related with this problem it would be also interesting to evaluate the anomalous dimensions for fermion composites, $\bar{\psi}\psi$, $\bar{\psi}\tau\psi$, \mathcal{O}_s , \mathcal{O}_p and so on at the fixed points. RG approach is just suitable for such purposes.

At the last of this section we would like to stress the advantageous points of the NPRG analyses, specially in comparison with the SD approaches. First it is necessary in the SD approach to examine the generalized models in order to explore the whole phase diagram at last. Whenever we derive the gap equations, we must assume the broken symmetries apriori and try some appropriate order parameters. Also we have to solve the coupled gap equations for them in the case that there are several non-trivial phases. For QED₃ we take care of the parity even mass and the parity odd mass. Therefore the equations become rather complicated to be solved even numerically. Thus the analysis will be rapidly harder in performing overall survey of the phase structures and in improving the approximations. In sharp contrast with such difficult situations, we realize through the above RG analysis that the NPRG method enables us to explore the whole phase diagrams quite easily. Moreover it is not necessary to care about the broken symmetries.

§4. Broken symmetries

So far it has been seen that the RG flows of the four-fermi couplings clarify the phase boundaries and also the fixed points for QED₃ and its generalizations. In this section we discuss the way to find out the broken symmetry in each phase. Our strategy is as follows. The theories belonging to the same phase, or the same universality class, are supposed to have the common symmetry. Therefore if we are able to find a much simpler model belonging to the phase concerned, then we may well examine the dynamical symmetry breaking in this model instead of the general theories. In our case the RG flows show that the three phase structure remains for the pure four-fermi theories ($e^2 = 0$) and also that both of the chirally broken and the parity broken phases are connected to those in the whole three dimensional theory space. So we may examine the broken symmetries by using the four-fermi theories.

*) In ref. 10), however, the four-fermi interactions added to QED₃ are not invariant under the $U(2N)$ symmetry, which is in contrast to our analysis.

For the explicit calculation we adopt the so-called auxiliary fields method. We shall introduce the following two auxiliary fields in the bare four-fermi theories:

$$\phi = -\left(G_{\text{P}} + \frac{G_{\text{S}}}{2N}\right)(\bar{\chi}_I \chi^I), \quad M_J^I + \frac{G_{\text{S}}}{G_{\text{S}} + 2NG_{\text{P}}} \delta_J^I \phi = -G_{\text{S}}(\bar{\chi}_J \chi^I). \quad (4.1)$$

Note that M is a traceless hermitian matrix field. The bare lagrangian is then rewritten as

$$\mathcal{L}_b = \bar{\chi} [\not{\partial} + M + \phi] \chi + \frac{1}{2G_{\text{S}}} \text{tr } M^2 + \frac{N}{G_{\text{S}} + 2NG_{\text{P}}} \phi^2. \quad (4.2)$$

To obtain an effective potential, we calculate only a one-loop correction for the fermions. Although this simple approximation is much rougher than the non-ladder one adopted in the section 3.4, it would be competent for our present purpose. Our effective potential becomes then

$$V_{\text{eff}}(M, \phi) = \frac{1}{2G_{\text{S}}} \text{tr } M^2 + \frac{N}{G_{\text{S}} + 2NG_{\text{P}}} \phi^2 - \int \frac{d^3 p}{(2\pi)^3} \ln \det [i\not{p} + M + \phi] \quad (4.3)$$

All we have to do is to search for the absolute minimum of this effective potential.

We here investigate only the case of $N = 1$ for the sake of simplicity. The matrix field M may be restricted as

$$M = \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix}. \quad (4.4)$$

After evaluating the last term in Eq. (4.3), we obtain

$$V_{\text{eff}}(d, \phi) = \frac{d^2}{G_{\text{S}}} + \frac{\phi^2}{G_{\text{S}} + 2G_{\text{P}}} - f(-d + \phi) - f(d + \phi), \quad (4.5)$$

where

$$f(x) = \frac{1}{6\pi^2} \left[\ln(1 + x^2) + 2x^2 - 2x^3 \arctan\left(\frac{1}{x}\right) \right]. \quad (4.6)$$

$f(x)$ behaves as $x^2/2\pi^2$ at $x \sim 0$ and $\ln|x|/3\pi^2$ at $x \sim \pm\infty$.

Let us consider the broken symmetry from the effective potential. If the field $d(\phi)$ acquires a vacuum expectation value, then the chiral (parity) symmetry is broken down. $f(-d + \phi) + f(d + \phi)$ reaches its maximal value when $-d + \phi = d + \phi$ or $-d + \phi = -(d + \phi)$, *i.e.* $\phi = 0$ or $d = 0$. Therefore the effective potential minimum must be on the lines, $\phi = 0$ or $d = 0$, which means these symmetries are never broken down simultaneously. Taking account of the first two terms in Eq.(4.5), the chiral (parity) symmetry breaking is impossible if G_{P} is positive (negative). When we make gap equations by differentiating the effective potential with respect to each field, we find if $G_{\text{P}} > 0$ and $G_{\text{S}} + 2G_{\text{P}} > \pi^2$, then the parity symmetry is broken down, and if $G_{\text{P}} < 0$ and $G_{\text{S}} > \pi^2$, then the chiral symmetry is broken down.

§5. Conclusions

We investigated the phase structure of dynamical symmetry breaking in QED_3 with N four-component massless fermions in the NPRG framework. The RG flows in the three dimensional theory space spanned by (e^2, G_s, G_p) were explicitly analyzed in the primitive approximation scheme, however, exceeding the ladder one. The beta functions for these couplings were gauge parameter independent. In this RG analysis the theories with the general four-fermi interactions invariant under the symmetries of QED_3 , $U(2N)$ and parity, were also automatically incorporated.

From the RG flows we found that the theory space was divided into three phases for $N > N_{\text{cr}}$, while the symmetric phase disappeared for $N < N_{\text{cr}}$. The critical flavor number was numerically estimated as $N_{\text{cr}} \simeq 3.1$. The phases were classified into the chiral symmetry broken, the parity broken and the scale invariant one. We elucidated the broken symmetries in each phase by examining the four-fermi theory belonging to the phase. Also the tri-critical fixed point was found to exist at the edge between the chirally broken phase and the parity broken one.

It was also seen within this approximation that the parity symmetry never be spontaneously broken in QED_3 with four-component fermions. This result was consistent with the Vafa-Witten theorem. For $N < N_{\text{cr}}$ the chiral symmetry was broken irrespective of the gauge coupling, while the critical gauge coupling dividing into chirally broken and unbroken phases was found to exist for $N > N_{\text{cr}}$. Since only the asymptotically free theories are examined in these analyses, these were also consistent with the known results obtained by solving the SD equations and by the lattice simulations

Through these analyses it was demonstrated that the NPRG offered us powerful and simple methods to explore the phase structures. The approximation scheme could be systematically improved in principle. However it has been the hard problem to maintain the gauge invariance non-perturbatively in the ERG method. Indeed our treatment for the gauge beta function was quite poor. For N not large the results were not totally confidential because the fixed points appears at the strongly coupled region. It should be hasten to develop the ERG framework for the gauge theories.

As final remarks we would like to mention other possible applications of such a NPRG approach. The similar dynamical phenomena to QED_3 has been known to occur also in four dimensions. If we consider QCD with appropriately many flavors, then the IR fixed point is found to exist²⁹⁾. There must be the critical number of flavors where the chiral symmetry breaking start to occur³⁰⁾. The essential mechanism of this transition may be understood by disappearance of the fixed point. Another example may be found in the high density

QCD. Recently the study of QCD with finite density has been revived and it is also believed that the so-called color superconducting phase exists³¹⁾. Actually most studies to these phenomena have been done by considering the SD equations. Most probably the NPRG will be useful also in these area.

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